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Large Perturbations of Elliptic Operators and the Solvability of the L^p Dirichlet Problem

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In this article, we investigate the solvability of the L^p Dirichlet problem for an elliptic operator, in divergence form, with bounded measurable coefficients. To be specific, assume that we have two such operators in the unit ball of R^n. Suppose that we know that the L^p Dirichlet problem is solvable for the first operator. We derive simple, easy to check criteria on the difference between the coefficients of the two operators which imply that the second operator has its L^p Dirichlet problem solvable as well. We emphasize that the value of p is the same for both operators and the criteria on the difference between the two coefficients allow for "large" perturbations (the definition of the notion of "large" perturbation is given quite precisely).

1. INTRODUCTION

In this article we study the solvability of the Dirichlet problem for elliptic equations with non-smooth coefficients when the boundary data belongs to some L^p class. This is a subject which has been investigated in the past, with many interesting results proven. The ones most relevant to our work here are perturbation theorems. That is, we are given two elliptic operators L_0 and L_1 and we seek easily checked criteria on the difference between the coefficients of these operators so that the solvability of the L^p Dirichlet problem for L_0 implies its solvability for L_1.

We first recall two such perturbation theorems. According to a result of Dahlberg [1], if the difference between the coefficients of L_0 and L_1 is sufficiently small in an appropriate norm (defined by a certain Carleson condition), then for fixed p the solvability of the L^p Dirichlet problem for L_0 implies that for the same value of p, the L^p Dirichlet problem has a solution for L_1. Next, if the disagreement between the coefficients is

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required to satisfy the same Carleson norm condition as in Dahlberg's theorem, only the value of the norm is no longer assumed to be small, then we can only conclude that the L^p Dirichlet problem's solvability for L_0 implies that the L^q Dirichlet problem is solvable for L_1 for some $q > p$. (See Fefferman *et al.* [2]). The natural question which remains is to find "the right way" of norming the difference between the coefficients of the two operators which guarantees the preservation of the solvability of the L^p Dirichlet problem for a fixed p (not two different values of p) but where we do not need to require the smallness of the norm. This article has as its main goal the study of this question. We obtain here the first results on the preservation of solvability of the L^p Dirichlet problem where the difference between the coefficients of L_0 and L_1 need not be uniformly small near the boundary. As an example of a situation where our results apply, suppose we consider two elliptic operators (L_0 and L_1) in the unit ball $B \subseteq \mathbb{R}^n$ whose coefficients are identical outside a non-tangential cone with vertex at some point on the unit sphere. Then we show that if the L^p Dirichlet problem is solvable for L_0 then the same is true for L_1 provided $p \geqslant (n-1)/(n-2)$, $n \geqslant 3$. We also study a question related to the ones considered above, namely that of determining which operators have their L^p Dirichlet problem solvable for all values of p, $1 < p < \infty$. As another application of our results, we examine the operators in $\mathbb{R}^n_+ = \{(x, t) \mid x \in \mathbb{R}^{n-1}, t > 0\}$ whose coefficients are homogeneous of degree 0, and smooth on S^{n-1}. This, of course, creates exactly one discontinuity on the boundary at the origin. Roughly speaking, the corresponding operators do not have their L^p Dirichlet problems solvable in the full range $1 < p < \infty$. If, however, the notion of homogeneity is taken with respect to the parabolic dilations $(x, t) \to (\delta x, \delta^\alpha t)$, $\alpha > 1$ where $\delta > 0$, then the operators whose coefficients are parabolically homogeneous of degree 0 "tend" to have their L^p Dirichlet problem solvable in the full range $1 \leqslant p < \infty$! Below, we are more precise about the exact statements of our results, as well as some important background which preceded our work. We also set the appropriate notations, and make the required definitions.

To be precise, we consider operators L of the form

$$Lu = \operatorname{div}(A \nabla u) \qquad \text{in} \quad D \subseteq \mathbb{R}^n,$$

where D is a smooth bounded domain (usually we shall take $D = B$, the unit ball of $\mathbb{R}^n$) and where $A(x)$ is a real bounded measurable $n \times n$ matrix valued function with

$$\begin{gathered} {}^tA(x) = A(x), \qquad \text{and} \qquad A(x) = (a_{ij}(x)) \\ \text{with} \hspace{20em} (\sharp) \\ \Lambda^{-1} |\xi|^2 \leqslant a_{ij}\xi_i\xi_j \leqslant \Lambda\, |\xi|^2 \qquad \text{for all} \quad x \in D \text{ and } \xi \in \mathbb{R}^n. \end{gathered}$$

Here the positive constant Λ is called the ellipticity constant of L. Now let us say exactly what we mean by "the L^p Dirichlet problem for L."

We say that $u(x)$ defined for $x \in D$ is a solution to $Lu = 0$ provided $u \in H^1_{\text{loc}}(D)$ and

$$\int_D a_{ij}\, \partial_i u\, \partial_j \varphi\, dz = 0 \qquad \text{for all} \quad \varphi \in C_0^\infty(D).$$

For each $x \in \partial D$, let us choose a right circular cone $\Gamma(x) \subseteq D$ with height and aperture fixed, with vertex at x, and oriented with its axis in the interior normal direction. We define the non-tangential maximal function of a function u in D by

$$u^*(x) = \sup_{z \in \Gamma(x)} |u(z)| \qquad \text{for} \quad x \in \partial D.$$

With this standard setup, we say that the L^p Dirichlet problem is solvable for the operator L if and only if to each $f \in L^p(d\sigma)$ (here $d\sigma$ is the surface area measure on the boundary of D) there corresponds a function u in D so that

1. $Lu = 0$ in D
2. $\lim_{z \to x,\, z \in \Gamma(x)} u(z) = f(x)$ for $(d\sigma)$ a.e. $x \in \partial D$, and
3. $\|u^*\|_{L^p(d\sigma)} \leqslant C\, \|f\|_{L^p(d\sigma)}$, if $p > 1$, or $\sigma\{u^* > \alpha\} \leqslant (C/\alpha)\, \|f\|_{L^1(d\sigma)}$ for all $\alpha > 0$, if $p = 1$.

As stated above, there are a number of important results concerning the solvability of the L^p Dirichlet problem in the literature. Let us choose four of these which are most relevant to our work, and state them here:

(i) (Caffarelli *et al.* [3]) There exist elliptic operators in the class defined by $\sharp$ (i.e., divergence form, with bounded measurable coefficients) such that the associated L^p Dirichlet problem is not solvable for any value of $p \geqslant 1$.

Thus, we must find those operators for which we can solve this problem. An important result in this direction is

(ii) (Fabes *et al.* [4]) Suppose L is an elliptic operator in B, $Lu = \operatorname{div}(A\, \nabla u)$, where A is continuous on $\bar{B}$, and set

$$\omega(\delta) = \sup_{\substack{x \in \partial B,\, y \in \bar{B} \\ |x - y| \leqslant \delta}} |A(x) - A(y)|.$$

Suppose $\int_0^1 \omega^2(\delta)\, d\delta/\delta < \infty$. Then the L^p Dirichlet problem is solvable for L in the full range $1 < p < \infty$.

In this last theorem (ii), it is crucial that the coefficients of L be continuous up to the boundary of B. Below, we state two perturbation results which apply in case the coefficients are discontinuous. We set the following notation:

Let $L_i u = \operatorname{div}(A_i \nabla u)$, $i = 0, 1$, be two elliptic operators in B with coefficients satisfying $\sharp$ above. For $z \in B$, set

$$a(z) = \operatorname*{ess\,sup}_{y \in B(z,\, \delta(z)/2)} |A_0(y) - A_1(y)|,$$

where $\delta(z) = 1 - |z|$ (in general $\delta(z)$ denotes the distance to the boundary of a domain). Then we have

(iii) (Theorem on small perturbations (Dahlberg [1])) Suppose L_i, $i = 0, 1$, are as above and the measure $a^2\, dz/\delta$ is a Carleson measure in B of vanishing trace. Then if the L^p Dirichlet problem is solvable for L_0, it is also solvable for L_1, provided $1 < p < \infty$.

(We recall that a positive measure μ in B is said to be a Carleson measure provided $\mu(B(x; r) \cap B) \leqslant Cr^{n-1}$ for all $x \in \partial B$ and $0 < r < 1$. We say that the Carleson measure μ has vanishing trace provided $\mu(B(x; r) \cap B)/r^{n-1} \to 0$ as $r \to 0$ uniformly in $x \in \partial B$.)

In case the measure $a^2\, dz/\delta$ is a Carleson measure, but does not have small Carleson norm (or vanishing trace), we can still say something:

(iv) (Theorem on large perturbations (Fefferman *et al.* [2])) Let L_0, L_1, and a be defined as above, and assume merely that $a^2\, dz/\delta$ is a Carleson measure in B. Then if the L^p Dirichlet problem is solvable for L_0, this implies that the L^q Dirichlet problem is solvable for L_1 for some $q < \infty$.

Thus, if, in the exact sense defined by the Carleson conditions above, we have a large perturbation, then the existence of a range of exponents p for which the L^p Dirichlet problem is solvable is inherited from L_0 to L_1. However, the exact value of p may change.

We find a condition on $a(z)$ similar to the requirement that $a^2\, dz/\delta$ be a Carleson measure which would guarantee that the solvability of the L^p Dirichlet problem for L_0 implies its solvability for L_1, but without imposing any smallness restriction on the relevant norm of $a(z)$.

It turned out that there was an obvious candidate for such a condition which appeared in a crucial way in the solution of Dahlberg's Conjecture (Theorem (iv) above).

This is the assumption that

$$\int_{\Gamma(x)} a^2 \frac{dz}{\delta^n} \leqslant C \qquad \text{for all} \quad x \in \partial B.$$

There was some strong evidence to indicate that even if C is large, condition (A) would imply the preservation of the solvability of the L^p Dirichlet problem, for $1 < p < \infty$. Unfortunately, as was shown in [2], this is not the case. Here, we show that if we replace the non-tangential condition (A) by an analogous tangential one, then this new condition does in fact imply the preservation of the solvability of the L^p Dirichlet problem. To be specific, in analogy with the Littlewood–Paley theory, where one studies the g_λ^* operator in addition to the area integral, we introduce

$$g_\lambda(a)(x) = \left(\int_B a^2(z) \left(\frac{\delta(z)}{|z-x|} \right)^{\lambda(n-1)} \frac{dz}{\delta^n(z)} \right)^{1/2} \qquad \text{for} \quad x \in \partial B.$$

In this article we prove the following theorem in $\mathbb{R}^n$, $n \geqslant 3$:

THEOREM. *There exists a λ depending only on the dimension n so that if the L^p Dirichlet problem is solvable for L_0 and if $g_\lambda(a) \in L^\infty(S^{n-1})$, then the L^p Dirichlet problem is also solvable for L_1.*

Here p is a fixed exponent satisfying $1 < p < \infty$, and we may take $\lambda = (n-2)/(n-1)$. Also note the main point that there is no restriction on the size of the L^∞ norm of $g_\lambda(a)$ in the assumption of the theorem above.

In the special case where we consider perturbations of the Laplacian, we also obtain large perturbation extension of the theorem of Fabes *et al.* ((ii) above). This corresponds to considering the conditions $g_\lambda(a) \in L^\infty$, where $\lambda > 1$. In fact, we show that if we consider operators whose coefficients lie in some fixed uniform neighborhood of the identity matrix, we may take $\lambda > 1$ in the theorem above.

In addition, we obtain another type of large perturbation theorem stated in terms of Carleson measures rather than L^∞ norms. Suppose L_0 and L_1 are elliptic operators in B with coefficients $A_0(z)$ and $A_1(z)$, respectively, and, as above, set $A(z) = \operatorname{ess\,sup}_{y \in B(z;\, \delta(z)/2)} |A_0(y) - A_1(y)|$. Then we prove the following:

THEOREM. *Let $1 < p < \infty$, and assume that the L^p Dirichlet problem is solvable for L_0. Assume also that*

$$\frac{1}{\sigma(\Delta)} \int_{S(\Delta)} a^2(z) \left(\frac{d}{\delta(z)} \right)^{p'} \frac{dz}{\delta} \leqslant C \qquad (\sim)$$

for every surface ball $\Delta = \Delta(x; d) \subseteq S^{n-1}$. Then the L^p Dirichlet problem is also solvable for L_1.

The proof of this theorem and also of the one involving the condition $g_\lambda(a) \in L^\infty$ is proven by an interpolation argument very much analogous to the Marcinkiewicz interpolation theorem. In our case we proceed by splitting up perturbations into large and small parts, rather than functions. (This method may be of interest in its own right.) We begin by proving an analogous result for certain quantities related to $g_\lambda(a)$ and the quantity appearing in ($\sim$). To be specific, we first prove that we get preservation of the solvability of the L^p Dirichlet problem under the assumption

$$g_\lambda^{(1)}(a) \in L^\infty \qquad \text{or} \qquad \left\| \left(\int_{\Gamma^d(x)} \left[a(z) \left(\frac{d}{\delta} \right) \right]^2 \frac{dz}{\delta^n} \right)^{1/2} \right\|_{L^{p',1}(d\sigma/\sigma(\Delta);\, \Delta)} \leqslant C$$

for all surface balls $\Delta \subseteq \partial B$. (Here d denotes the radius of Δ, $\Gamma^d(x)$ is the cone with vertex at x and truncated at height d, and $L^{p',1}$ is the Lorentz space.) The definition of $g_\lambda^{(1)}(a)$ is

$$g_\lambda^{(1)}(a)(x) = \int_B a(z) \left(\frac{\delta(z)}{|x-z|} \right)^{\lambda(n-1)} \frac{dz}{\delta^n},$$

a linear version of g_λ. We prove the perturbation theorems first for these functionals because working with them, setting $L_t = (1-t) L_0 + tL_1$, and $Q(t)$ the $B^p(d\sigma)$ norm of the harmonic measure ω_{L_t}, we may derive $|Q'(t)| \leqslant CQ(t)$. (This does not seem possible when we work under the more satisfactory assumptions $g_\lambda(a) \in L^\infty$ or $(1/\sigma(\Delta)) \int_{S(\Delta)} a^2(d/\delta)^{p'} dz/\delta \leqslant C$.) We then pass from the $g_\lambda^{(1)}$ assumption to the g_λ one by a process of splitting the perturbation $L_0 \to L_1$ into a "small part" $L_0 \to M$, to which Dahlberg's Small Perturbation Theorem applies and a "large part" $M \to L_1$ for which $g_\lambda^{(1)}(a) \in L^\infty$. Similar comments apply to the passage from the mixed norm Carleson condition to the more familiar one ($\sim$). We should also remark here that the condition $g_\lambda^{(1)}(a) \in L^\infty$ for certain λ has the additional meaning of guaranteeing the preservation of the solvability of the L^1 Dirichlet problem, and this is not shared by the more familiar quadratic condition $g_\lambda(a) \in L^\infty$.

This paper is organized as follows: Section 2 contains a pair of lemmas on the doubling properties of the harmonic measure associated to an elliptic operator. The first lemma, which considers arbitrary elliptic operators, is due to Kenig and Pipher [5]. The second lemma concerns operators whose coefficients are uniformly close to those of the Laplacian. In the more difficult case of non-divergence form operators, it is due to Santiago Marin-Malavé [6]. Even for divergence form operators, we have been unable to locate a proof of this result in the literature, and we are grateful to Kenig for providing us with the simple argument given below.

In Section 3, we prove preliminary versions of large perturbation theorems (assuming $g_\lambda^{(1)}(a) \in L^\infty$ or a mixed norm Carleson condition) by Dahlberg's method of differential inequalities. That is, given operators L_0 and L_1, we set $L_t = (1-t)L_0 + tL_1$, $t \in [0, 1]$, and let ω_t denote the harmonic measure associated to L_t. Then under these assumptions on $a(z)$, we derive, in Section 3, estimates of the form $|Q'(t)| \leqslant CQ(t)$, where $Q(t) = \|\omega_t\|_{B^p(d\sigma)}$, and this allows us to eliminate any assumption of smallness of the perturbation.

Section 4 is devoted to the interpolation arguments that are necessary in order to pass from assumptions such as $g_\lambda^{(1)}(a) \in L^\infty$ to ones like $g_\lambda(a) \in L^\infty$.

Finally, in Section 5, we give the applications mentioned above, as well as some examples which may help the reader understand the results.

Finally, we take this opportunity to thank Professor E. Fabes for several interesting discussions, and, in particular, for pointing out the work of Marin-Malavé to us.

2. Doubling Properties of Harmonic Measure

In this article we encounter several times the expression $\omega(\Delta)/\omega(\tilde{\Delta})$, where Δ denotes the harmonic measure associated with an elliptic operator B and $\Delta \subseteq S^{n-1}$ is a surface ball while $\tilde{\Delta}$ is a given concentric enlargement of Δ. We need to bound this ratio in terms of the surface measure of Δ and $\tilde{\Delta}$ and this is done below in a pair of lemmas. The first of the lemmas tells us what can be said about the ratio $\omega(\Delta)/\omega(\tilde{\Delta})$ for the harmonic measure of a general elliptic operator.

Lemma 2.1. (Kenig and Pipher [5]) *Let $Lu = \operatorname{div}(A \nabla u)$ be an elliptic operator defined in the unit ball $B \subset \mathbb{R}^n$, $n > 2$. Let $\Delta \subset \tilde{\Delta}$ be two surface balls on S^{n-1}. Then we have*

$$\frac{\omega(\Delta)}{\omega(\tilde{\Delta})} \leqslant C \left[\frac{\sigma(\Delta)}{\sigma(\tilde{\Delta})}\right]^{(n-2)/(n-1)},$$

where the constant C depends only on the ellipticity constant L and the dimension n.

Proof. Let $\tilde{\Delta} = B(x_0; r) \cap S^{n-1}$, and put $A = B(x_0; 2r) \cap B$. By a standard application of the Comparison Theorem, we see that if ω' denotes the harmonic measure of the operator L restricted to A (evaluated at the "center" of A), then

$$\frac{\omega(\Delta)}{\omega(\tilde{\Delta})} \leqslant C \frac{\omega'(\Delta)}{\omega'(\tilde{\Delta})},$$

where C depends only on the ellipticity constant of L. Now we dilate the region A to get a region A_1 of diameter one, and consider the operator L_1 in A_1 obtained by a dilation of the coefficients of L in A. Letting ω_1 be the harmonic measure of L_1 in A_1, and Δ_1 and $\tilde{\Delta}_1$ denote the dilates of $\Delta, \tilde{\Delta}$ to the scale of A_1, we have

$$\omega'(\Delta) = \omega_1(\Delta_1), \qquad \omega'(\tilde{\Delta}) = \omega_1(\tilde{\Delta}_1) \geqslant c,$$

where c depends only on the ellipticity constant of L. Our lemma is then reduced to the estimate

$$\omega_1(\Delta_1) \leqslant C\sigma(\Delta_1)^{(n-2)/(n-1)}.$$

This follows immediately from the standard estimate for harmonic measure in terms of the Green's function,

$$\omega_1(\Delta_1) \leqslant C[\text{diameter}(\Delta_1)]^{n-2}\, G(z_{\Delta_1}) \leqslant C'\sigma(\Delta_1)^{(n-2)/(n-1)},$$

where z_{Δ_1} is the point of A_1 above the center point of Δ_1 by a distance equal to diameter of Δ_1, and we have used the standard pointwise estimate for the Green's function. This proves the lemma.

Next we consider an estimate analogous to the one given in the previous lemma when the operator has coefficients close to those of the Laplacian. In the more difficult case of non-divergence form operators this result has been obtained by Marin-Malavé [6]. The simple proof here in our divergence form setting has been given by Kenig:

LEMMA 2.2. *Let $\varepsilon > 0$ be given. Then there exists a $\delta > 0$ and a constant C depending only on ε and δ so that if $Lu = \operatorname{div}(A\,\nabla u)$ and if $|A(z) - I| < \delta$ for all $z \in B$, then*

$$\frac{\omega(\Delta)}{\omega(\tilde{\Delta})} \leqslant C\left[\frac{\sigma(\Delta)}{\sigma(\tilde{\Delta})}\right]^{1-\varepsilon}$$

for all surface balls $\Delta \subseteq \tilde{\Delta}$ on S^{n-1}.

Proof. Using the Comparison Theorem and dilation invariance of our class of operators, just as in Lemma 2.1, it suffices to show that if Δ is a small surface ball on S^{n-1}, then under the assumption that $|A(z) - I| < \delta$

for all $z \in B$ we have $\omega(\Delta) \leqslant C\sigma(\Delta)^{1-\varepsilon}$ if $\delta > 0$ is sufficiently small. We consider the modified Green's function $\tilde{G}$ for the operator $Lu = \operatorname{div}(A \nabla u)$ defined by $L(\tilde{G}) = -\varphi$, $\tilde{G}|_{\partial B} = 0$, where $\varphi \geqslant 0$, $\varphi \in C_0^\infty(B(0; \frac{1}{10}))$. For small surface balls by the standard estimate on Green's functions,

$$\omega(\Delta) \leqslant C\rho^{n-2}\tilde{G}(z_\Delta),$$

where ρ is the diameter of Δ and z_Δ is the point $(1-\rho)x_0$, if x_0 is the center of Δ. We indicate below a proof that $\tilde{G}$ is a Lipschitz function on $\bar{B}$ of order $1-(n-1)\varepsilon$ if δ is sufficiently small. This completes the proof of the lemma since then we have

$$\omega(\Delta) \leqslant C'_\delta \rho^{n-2} \cdot \rho^{1-(n-1)\varepsilon} \leqslant C''_\delta \sigma(\Delta)^{1-\varepsilon}.$$

To show that $\tilde{G}$ is Lipschitz of order close to one up to the boundary we estimate $\nabla\tilde{G}$ as follows:

Write ∂_i as ΛR_i, where R_i is the ith Riesz transform, and Λ is Calderón's operator: $\Lambda f^\wedge(\xi) = |\xi| f^\wedge(\xi)$. Then

$$\Lambda R_i a_{ij} R_j \Lambda(\tilde{G}) = -\varphi.$$

Choose $p < \infty$ so large that by the Sobolev Embedding Theorem $\nabla\tilde{G} \in L^p(B)$ implies that $\tilde{G}$ is Lipschitz of order $1-\varepsilon(n-1)$, as desired. Then we show that $\tilde{G} \in H^{1,p}(B)$ (i.e., $\tilde{G}$ has one derivative in $L^p(B)$) by writing

$$\tilde{G} = \Lambda^{-1} T^{-1} \Lambda^{-1}(-\varphi),$$

where $T = R_i a_{ij} R_j$ and where we show that by assuming δ to be sufficiently small and $|A(z) - I| < \delta$ for all $z \in B$, then T is invertible on $L^p(B)$. Since $\Lambda^{-1}(-\varphi) \in L^p(\mathbb{R}^n)$ for p large enough, this invertibility implies the lemma. To see that T is invertible on L^p, we write $T = R_i(\delta_{ij} + \eta_{ij})R_j$, where $|\eta_{ij}|$ are made uniformly small. Since $R_i \delta_{ij} R_j = I$, the operator T is of the form $I + R_i \eta_{ij} R_j$, and as long as the operator $R_i \eta_{ij} R_j$ has a norm (acting on L^p) less than one, T is invertible. This happens when η_{ij} are uniformly small (how small depends on p). The proof of Lemma 2.2 is then complete.

In the next section we require an estimate of the kernel function corresponding to the operators of Lemma 2.2. This estimate follows from the same type of analysis that was done in this lemma.

COROLLARY. *Suppose $Lu = \operatorname{div}(A \nabla u)$ is an elliptic operator with bounded measurable coefficients in B. Suppose that for $z \in B$ and $x \in \partial B$,*

$K(z, x)$ denotes the kernel function for L. Then given $\varepsilon > 0$, there exists a $\delta > 0$ so that if $|A(z) - I| < \delta$ for all $z \in B$, then

$$K(z, x) \leqslant C_{\varepsilon,\delta} \sum_{k \geqslant 0} 2^{-k(1-\varepsilon)} \frac{\chi_{\tilde{\Delta}_z^k}(x)}{\omega(\tilde{\Delta}_z^k)},$$

where $\tilde{\Delta}_z^k = \Delta(z/|z|; 2^k\delta(z))$.

Proof. By the Comparison Theorem, and the definition of the kernel function as $(d\omega^z/d\omega^0)(x)$, $x \in \partial B$, we have

$$K(z, x) \leqslant C \frac{\omega^z(\Delta_{x,z})}{\omega(\tilde{\Delta}_z^k)},$$

provided $2^{k-1}\delta(z) \leqslant |z/|z| - x| < 2^k\delta(z)$ and $\Delta_{x,z} = \Delta(x, 2^{k-5}\delta(x))$. It therefore suffices to show that $\omega^z(\Delta_{x,z}) \leqslant C2^{-k(1-\varepsilon)}$ if $|A(z) - I| < \delta$ for all $z \in B$, and $\delta > 0$ is sufficiently small.

By rescaling, this is essentially equivalent to the following estimate: Assume $|A(z) - I| < \delta$ throughout B. We prove that if H is the northern hemisphere of S^{n-1}, then $\omega^z(H) \leqslant C\delta(z)^{1-\varepsilon}$ provided z is near the south pole. In fact, consider a function $\Phi \in C^\infty(\bar{B})$ and its restriction φ to S^{nb-1} satisfying $\Phi \geqslant 0$ in $\bar{B}$, $\varphi \geqslant 1$ on H, and $\varphi = 0$ in the southern quarter sphere. Then clearly if $Lv = 0$ in B, $v|_{\partial B} = \varphi$, it follows that

$$v(z) \geqslant \omega^z(H),$$

and we prove that $v(z) \leqslant C\delta(z)^{1-\varepsilon}$ for z near the south pole. In fact, setting $w = v - \Phi$, then

$$L(w) = L(v - \Phi) = -L(\Phi) \in H^{-1,p} \qquad \text{for all} \quad p < \infty.$$

Using exactly the same argument as in Lemma 2.2 above, we see that w is Lipschitz of order $1 - \varepsilon$ as long as $\delta > 0$ is small enough. This shows that v is also Lipschitz of order $1 - \varepsilon$ on $\bar{B}$ and finishes the proof of the corollary.

We are now prepared to prove the perturbation results which lead to the main theorems of this paper.

3. Some Basic Special Cases of Large Perturbation Theorems

In this section we begin by dealing with perturbations of elliptic operators defined by linear functionals of the difference of the coefficients of the operators, rather than the more familiar quadratic ones. These perturbations are the right ones to consider if we are concerned about the preservation of the solvability of the L^p Dirichlet problem for $p \geqslant 1$.

Let us make the appropriate definitions. Let $\Gamma_\gamma(x)$, $x \in S^{n-1}$, denote a right circular cone with vertex at x and aperture γ. Let $a(z)$ denote a positive function in B. Then, if $\lambda > 0$ and $\delta(z) = 1 - |z|$, set

$$S_\gamma^{(1)}(a)(x) = \int_{\Gamma_\gamma(x)} a(z) \frac{dz}{\delta^n(z)}$$

and

$$g_\lambda^{(1)}(a)(x) = \int_B a(z) \left(\frac{\delta(z)}{|z-x|} \right)^{\lambda(n-1)} \frac{dz}{\delta^n(z)}.$$

Then we have

THEOREM 3.1. *Let $L_i u = \operatorname{div}(A_i \nabla u)$, $i = 0, 1$, denote elliptic operators with bounded measurable coefficients in B. Suppose, for $z \in B$,*

$$a(z) = \operatorname*{ess\,sup}_{y \in B(z;\, \delta(z)/2)} |A_0(y) - A_1(y)|.$$

Then (i) *In dimension n, $n > 2$, if $g_\lambda^{(1)}(a) \in L^\infty$ for $\lambda = (n-2)/(n-1)$ and if the L^p Dirichlet problem is solvable for L_0, for some value of p $(1 \leqslant p < \infty)$, then the L^p Dirichlet problem is also solvable for L_1. (If $n = 2$, λ can be chosen depending only on the ellipticity constant of L_0, L_1 such that the same conclusion is true.)*

(ii) *If $n = 2$ and if L_0 and L_1 satisfy merely $S_\gamma^{(1)}(a) \in L^\infty$ for some $\gamma > 0$ then we cannot conclude that the L^p Dirichlet problem is solvable for L_1 for all p, $1 < p < \infty$, even if $L_0 = \Delta$.*

Proof. To prove part (i), we show that assuming $g_\gamma^{(1)}(a) \in L^\infty$, $\lambda = (n-2)/(n-1)$, and $\omega_0 \in B^p(d\sigma)$, then $\omega_1 \in B^p(d\sigma)$ where $1 < p < \infty$, ω_i denotes the harmonic measure associated to L_i and $\omega = k\, d\sigma \in B^p(d\sigma)$ means that for all surface balls $\Delta \subseteq S^{n-1}$

$$\left(\frac{1}{\sigma(\Delta)} \int_\Delta k^p \, d\sigma \right)^{1/p} \leqslant C_p \left(\frac{1}{\sigma(\Delta)} \int_\Delta k, d\sigma \right),$$

or, when $p = \infty$,

$$\operatorname*{ess\,sup}_{x \in \Delta} k(x) \leqslant C_\infty \frac{1}{\sigma(\Delta)} \int_\Delta k \, d\sigma.$$

Taking into account the well known fact that the L^p Dirichlet problem is solvable if and only if $\omega_L \in B^{p'}(d\sigma)$ $(1/p+1/p'=1,\ 1\leqslant p<\infty)$ this is enough to prove part (i).

We begin the proof following Dahlberg [1]: Define the operators $L_t = (1-t)L_0 + tL_1$ for $0 \leqslant t \leqslant 1$. Let L_t have harmonic measure ω_t and let the $B^p(d\sigma)$ norm of ω_t be denoted $Q(t)$. We define a quantity (as in [1]) which is "equivalent to $Q(t)$" and then prove that the derivative in t of this quantity is dominated by $Q(t)$ itself. This controls $Q(1)$ in terms of $Q(0)$ and finishes the proof. So, let Δ be a surface ball so that

$$\|k_1\|_{L^p(d\sigma/\sigma(\Delta);\,\Delta)} / \|k_1\|_{L^1(d\sigma/\sigma(\Delta);\,\Delta)} \geqslant \tfrac{1}{2}\,Q(1).$$

Set $\Delta = \Delta(x_0; r_0)$, and $A = B(x_0, 2r_0) \cap B$. According to the Comparison Theorem [7], if we now take all the operators L_t to be defined in the domain A and refer to the harmonic measures of these restrictions to A of L_t still by $\omega_t = k_t\, d\sigma$, then the B^p norm of ω_t and the ratios

$$\|k_t\|_{L^p(d\sigma/\sigma(\Delta))} / \|k_t\|_{L^1(d\sigma/\sigma(\Delta))}$$

are changed by at most a fixed factor depending only on ellipticity of the L_i. Henceforth in this proof all operators and harmonic measures are taken relative to the domain A. So we must estimate

$$\left(\frac{1}{\sigma(\Delta)}\int_\Delta k_1^p\, d\sigma\right)^{1/p} \Big/ \frac{1}{\sigma(\Delta)}\int k_1\, d\sigma,$$

and using the rescaling down to A, we have $\int_\Delta k_1\, d\sigma \geqslant c$, where c depends only on the ellipticity constants of L_0 and L_1. Take $f \geqslant 0$ a continuous function on ∂A which is supported in Δ with $\|f\|_{L^{p'}(d\sigma/\sigma(\Delta);\,\Delta)} = 1$, and we must then estimate

$$\int_\Delta f k_1\, d\sigma.$$

Define u_t and h_t by

$$L_t u_t = 0, \qquad u_t|_{\partial A} = f$$

and

$$L_t h_t = -\varphi \qquad h_t|_{\partial A} = 0,$$

where $\varphi \geqslant 0$ in A, $\varphi \in C_0^\infty(A)$, $\int_A \varphi = 1$ and φ is supported in $B(\bar{z}; r_1/100)$ where $\bar{z} = (1 - \frac{9}{5} r_0)x_0$.

By Harnack's principle, it suffices to estimate

$$\int_A u_1(z)\,\varphi(z)\,dz.$$

A calculation by Dahlberg in [1] shows that if $\bullet$ denotes differentiation with respect to t, we have

$$\int_A u_t(z)\,\dot{\varphi}(z)\,dz = \int_A \varepsilon(z)\,\nabla u_t(z)\,\nabla h_t(z)\,dz,$$

where $\varepsilon(z) = A_1(z) - A_0(z)$. We now show that assuming $g_\lambda^{(1)}(a) \in L^\infty$, $\lambda = (n-2)/(n-1)$, then

$$\left| \int_A u_t(z)\,\dot{\varphi}(z)\,dz \right| \leqslant C\,\|g_\lambda^{(1)}(a)\|_{L^\infty}\,Q(t)$$

and, as mentioned above, this finishes the proof.

To make this estimate set $A^{(1)} = B(x_0; \frac{3}{2}r_0) \cap B$ and $A^{(2)} = A - A^{(1)}$. We estimate

$$\begin{aligned} &\int_{A^{(2)}} |\varepsilon(z)|\,|\nabla u_t(z)|\,|\nabla h_t(z)|\,dz \\ &\qquad \leqslant \|\varepsilon\|_\infty \left(\int_{A^{(1)}} |\nabla u_t(z)|^2\,dz \right)^{1/2} \left(\int_{A^{(2)}} |\nabla h_t(z)|^2\,dz \right)^{1/2}. \end{aligned}$$

An easy Cacciopolli estimate (and the fact that u vanishes on $\partial B(x_0; 2r_0) \cap A$) yields

$$\left(\int_{A^{(2)}} |\nabla u_t(z)|^2\,dz \right)^{1/2} \leqslant C r_0^{-1} \left(\int_{A^{(2)}} u_t^2(z)\,dz \right)^{1/2} \leqslant C r_0^{-1} u_t(\bar{z})\,r_0^{n/2},$$

where the last estimate uses the Harnack principle up to the boundary of $A^{(2)}$ ([7]).

Also standard estimates on Green's functions show that

$$\int_{A^{(2)}} |\nabla h_t(z)|^2\,dz \leqslant C r_0^{2-n},$$

and putting this together gives

$$\begin{aligned} \int_{A^{(2)}} |\varepsilon|\,|\nabla u_t|\,|\nabla h_t|\,dz &\leqslant C u_t(\bar{z}) \leqslant C \int_\Delta f k_t \\ &\leqslant C\sigma(\Delta)\,\|f\|_{L^p(d\sigma/\sigma(\Delta))} \cdot \|k_t\|_{L^{p'}(d\sigma/\sigma(\Delta))} \\ &\leqslant C\sigma(\Delta)\,Q(t)\,\frac{1}{\sigma(\Delta)} \int_\Delta k_t\,d\sigma \\ &\leqslant CQ(t). \end{aligned}$$

Now we estimate $\int_{A^{(1)}} |\varepsilon(z)|\, |\nabla u_t(z)|\, |\nabla h_t(z)|\, dz$ as follows:

$$\begin{aligned}
&\int_{A^{(1)}} \varepsilon(z)\, |\nabla u_t(z)|\, |\nabla h_t(z)|\, dz \\
&\quad\leqslant C \int_{A^{(1)}} \int_{y \in B(z;\, \delta(z)/2)} \varepsilon(y)\, |\nabla u_t(y)|\, |\nabla h_t(y)|\, dy \frac{dz}{\delta^n(z)} \\
&\quad\leqslant C \int_{A^{(1)}} a(z) \left(\int_{y \in B(z;\, \delta(z)/2)} |\nabla u_t(y)|^2\, dy \right)^{1/2} \\
&\qquad \times \left(\int_{y \in B(z;\, \delta(z)/2)} |\nabla h_t(y)|^2\, dy \right)^{1/2} \frac{dz}{\delta^n(z)}.
\end{aligned}$$

By Cacciopolli's inequality

$$\begin{aligned}
\int_{B(z;\, \delta(z)/2)} |\nabla u_t(y)|^2\, dy &\leqslant C\delta^{-2}(z) \int_{B(z;\, (3/4).\, \delta(z))} u_t^2(y)\, dy \\
&\leqslant C\delta^{n-2}(z)\, u_t^2(z)
\end{aligned}$$

using Harnack's principle, and a similar estimate holds for h_t as well. This shows that

$$\int_{A^{(1)}} \varepsilon(z)\, |\nabla u_t(z)|\, |\nabla h_t(z)|\, dz \leqslant C \int_{A^{(1)}} a(z)\, \delta^{-2}(z)\, u_t(z)\, h_t(z)\, dz.$$

But standard estimates on the Green's function give

$$h_t(z) \leqslant \delta^{2-n}(z)\, \omega_t(\Delta_z),$$

where the surface ball Δ_z is defined by $\Delta_z = \Delta(z/|z|;\, \delta(z))$. Substituting this in the integral above, we need to estimate

$$\int_{A^{(1)}} a(z)\, u_t(z)\, \omega_t(\Delta_z) \frac{dz}{\delta^n(z)}.$$

Let $K_t(z, x)$ denote the kernel function for L_t in A. Then

$$u_t(z) = \int_{\partial A} K_t(z, x)\, f(x)\, d\omega_t(x).$$

The basic estimate of Caffarelli *et al.* [7] for K_t is

$$K_t(z, x) \leqslant \sum_{k \geqslant 0} 2^{-k\eta} \frac{\chi_{\tilde{\Delta}_z^k}(x)}{\omega_t(\tilde{\Delta}_z^k)},$$

where $\tilde{\Delta}_z^k$ is the surface ball with center $z/|z|$ and radius $2^k\delta(z)$, and where $\eta > 0$ depends only on the ellipticity of L_t. Thus, we must now estimate

$$\int_{z \in A(1)} a(z) \int_{x \in \Delta} f(x) \sum_k 2^{-\Delta k\eta} \frac{\chi_{\tilde{\Delta}_z^k}(x)}{\omega_t(\tilde{\Delta}_z^k)} d\omega_t(x)\, \omega_t(\Delta_z) \frac{dz}{\delta^n(z)}$$

$$= \int_\Delta f(x) \left(\int_{z \in A^{(1)}} a(z) \sum_k 2^{-k\eta} \chi_{\tilde{\Delta}_z^k}(x) \left[\frac{\omega_t(\Delta_z)}{\omega_t(\tilde{\Delta}_z^k)} \right] \frac{dz}{\delta^n(z)} \right) k_t(x)\, d\sigma(x).$$

At this point, we apply Lemma 2.1 which tells us that

$$\frac{\omega_t(\Delta_z)}{\omega_t(\tilde{\Delta}_z^k)} \leqslant C 2^{-k(n-2)},$$

where C depends only on the ellipticity of L_t. This tells us that the expression above is dominated by

$$C \int_\Delta f(x)\, k_t(x) \int_{z \in A^{(1)}} a(z) \sum_z 2^{-k(n-2)} \chi_{\tilde{\Delta}_z^k}(x) \frac{dz}{\delta^n(z)}\, d\sigma(x).$$

Observe that for fixed x, the sum $\sum_k 2^{-k(n-2)} \chi_{\tilde{\Delta}_z^k}(x)$ is $\sum 2^{-k(n-2)}$, where the sum is over those k so that $|x - z'| < 2^k\delta(z)$ $(z' = z/|z|)$; that is, $2^k > |x - z'|/\delta(z)$ so substituting this into the last integral, we see that

$$\begin{aligned} &\int a(z) \sum 2^{-k(n-2)} \chi_{\tilde{\Delta}_z^k}(x) \frac{dz}{\delta^n(z)} \\ &\quad \leqslant C \int a(z) \left(\frac{\delta(z)}{\delta(z) + |x - z'|} \right)^{n-2} \frac{dz}{\delta^n(z)} \\ &\quad \leqslant C \int a(z) \left(\frac{\delta(z)}{|x - z|} \right)^{\lambda(n-1)} \frac{dz}{\delta^n(z)} = g_\lambda^{(1)}(a)(x) \end{aligned}$$

for $\lambda = (n-2)/(n-1)$. This shows that

$$\begin{aligned} \int_{A^{(1)}} \varepsilon(z)\, |\nabla u_t(z)|\, |\nabla h_t(z)|\, dz &\leqslant C \int_\Delta f(x)\, k_t(x)\, g_\lambda^{(1)}(a)(x)\, d\sigma(x) \\ &\leqslant C\, \|g_\lambda^{(1)}(a)\|_\infty\, \sigma(\Delta) \int_\Delta f(x)\, k_t(x) \frac{d\sigma}{\sigma(\Delta)} \\ &\leqslant C\sigma(\Delta)\, \|g_\lambda^{(1)}(a)\|_\infty\, \|f\|_{L^{p'}(d\sigma/\sigma(\Delta))}\, \|k_t\|_{L^p(d\sigma/\sigma(\Delta))} \\ &\leqslant C\, \|g_\lambda^{(1)}(a)\|_\infty\, \|k_t\|_{L^1(d\sigma)}\, Q(t) \\ &\leqslant C\, \|g_\lambda^{(1)}(a)\|_\infty\, Q(t), \end{aligned}$$

and this completes the proof of part (i).

Proof of part (ii). This is essentially already contained in [2]. We merely indicate a brief sketch of the proof, and refer the reader to [2] for the details.

The main idea is to look at examples of elliptic operators with homogeneous coefficients. We construct an example of an operator $L = \operatorname{div}(A\nabla)$ in D_+, the upper half disk of $\mathbb{R}^2$ (rather than in D), which has the property that $S_\gamma^{(1)}(a) \in L^\infty(\partial D_+)$ for γ small enough, where

$$a(z) = \operatorname*{ess\,sup}_{y \in B(z;\, \delta(z)/2)} |A(y) - I|,$$

but whose harmonic measure ω does *not* belong to all the classes $L^p(d\sigma)$ for all $p < \infty$.

To do this let $\Phi\colon \mathbb{R}_+^2 \mapsto \mathbb{R}_+^2$ be given by a smooth function, homogeneous of degree $\alpha + 1$ $(\alpha > -1)$,

$$\Phi(x, t) = |(x, t)|^\alpha \,(x, t).$$

Then if $\Delta u = 0$ in $\mathbb{R}_+^2$, it follows that $\operatorname{div}(B\nabla(u \circ \Phi)) = 0$ where $B(x, t)$ is smooth away from the origin, and homogeneous of degree 0. The harmonic measure of $\operatorname{div}(B\nabla)$ is homogeneous of degree α, and if $\alpha < 0$ it follows that the harmonic measure is not in every $L^p(d\sigma)$ class. The coefficients are constant on the real axis, so by a linear change of variables, we arrive at an elliptic operator $Lu = \operatorname{div}(A\,\nabla u)$ which has coefficient function $A(x, t)$ smooth away from the origin, homogeneous of degree 0, and such that $A(x, 0) = I$ for all $x \in \mathbb{R}$. Of course, we still have $\omega_L \notin \bigcap_{p<\infty} B^p(d\sigma)$.

As is shown in [2], $S_\gamma^{(1)}(a) \in L^\infty$ provided $A(x, t) \equiv I$ for those points on the unit circle close enough to the point $(0, 1)$ (γ depends on how close this is). Although this is not necessarily the case for the operator we have constructed, we can perturb L to get a new operator $\tilde{L}$ which provides the example we require to prove part (ii). To construct $\tilde{L}$ from L, simply make the coefficient of $\tilde{L}$ the same as that of L outside of a cone $\Gamma_\alpha(0)$ with vertex at the origin, and small aperture α, and inside $\Gamma_\alpha(0)$, set the coefficient of $\tilde{L}$ identically equal to the identity. Then as is shown in [2] we may consider $\tilde{L}$ a "small perturbation" of L measured in the appropriate Carleson norm. According to Dahlberg's small perturbation theorem, it follows that $\omega_L \notin B^p(d\sigma)$ implies $\omega_{\tilde{L}} \notin B^q(d\sigma)$ for some q large, (see [2]) provided the perturbation L to $\tilde{L}$ is small enough. Thus, the construction of $\tilde{L}$ proves part (ii) of Theorem 3.1.

In the next theorem, we investigate two cases where we may assume that $g_\lambda^{(1)}(a) \in L^\infty$ with $\lambda > 1$, and draw the desired conclusion about the solvability of the L^p Dirichlet problem for an operator.

THEOREM 3.2. (i) *Let* $L_i u = \operatorname{div}(A_i \nabla u)$, $i = 0, 1$, *be elliptic operators with bounded measurable coefficients in* B. *Define, as usual,* $a(z) = \operatorname{ess\,sup}_{y \in B(z;\, \delta(z)/2)} |A_0(y) - A_1(y)|$. *Then there exists* $\delta > 0$ *depending only on the ellipticity constants of* L_0 *and* L_1 *such that if*

$$\| g^{(1)}_{1+\delta}(a) \|_\infty < \varepsilon,$$

and $\varepsilon > 0$ *is sufficiently small (depending on ellipticity of the* L_i *and the* B^∞ *norm of* ω_0*) then the solvability of the* L^1 *Dirichlet problem for* L_0 *implies the solvability of the* L^1 *Dirichlet problem for* L_1.

(ii) *Let* $Lu = \operatorname{div}(A \nabla u)$ *be elliptic with bounded measurable coefficients in* B. *Let* $a(z) = \operatorname{ess\,sup}_{y \in B(z;\, \delta(z)/2)} |A(y) - I|$. *Then for any* δ *satisfying* $0 < \delta < n/(n-1)$, *there exists* $\varepsilon > 0$ *depending only on* δ *and* n *so that if* $a(z) < \varepsilon$ *for all* $z \in B$ *and* $g^{(1)}_{n/(n-1)-\delta}(a) \in L^\infty$, *then the* L_1 *Dirichlet problem is solvable for* L.

Proof. (i) We look at the proof of Theorem 3.1, taking $Q(t)$ to be the $B^\infty(d\sigma)$ norm of $L_t = (1-t)L_0 + tL_1$. We see from the proof there that essentially

$$|\dot{Q}(t)| \leqslant C \|a\|_\infty Q(t) + \int_\Delta f(x) \int_{z \in A^{(1)}} a(z) \left(\sum_k 2^{-k\eta} \chi_{\tilde{\Delta}^k_z}(x) \left[\frac{\omega_t(\Delta_z)}{\omega_t(\tilde{\Delta}^k_z)} \right] \right)$$
$$\times \frac{dz}{\delta^n(z)} k_t(x)\, d\sigma(x). \tag{$*$}$$

Here $\eta > 0$ depends only on the ellipticity of L_t. Observe now that

$$\begin{aligned} \frac{\omega_t(\Delta_z)}{\omega_t(\tilde{\Delta}^k_z)} &= \frac{1}{\omega_t(\tilde{\Delta}^k_z)} \int_{\tilde{\Delta}^k_z} k_t \chi_{\Delta_z}\, d\sigma(x) \\ &\leqslant \|k_t\|_{L^\infty(\tilde{\Delta}^k_z)} \frac{\sigma(\Delta_z)}{\omega_t(\tilde{\Delta}^k_z)} \\ &\leqslant \|k_t\|_{B^\infty(d\sigma)} \frac{\omega_t(\tilde{\Delta}^k_z)}{\sigma(\tilde{\Delta}^k_z)} \frac{\sigma(\Delta_z)}{\omega_t(\tilde{\Delta}^k_z)} \leqslant Q(t) \cdot 2^{-k(n-1)}. \end{aligned}$$

Substituting this in ($*$) above, we see that

$$\begin{aligned} |\dot{Q}(t)| &\leqslant C \|a\|_\infty Q(t) \\ &\quad + \left[\int_\Delta f(x) \int_{z \in A^{(1)}} a(z) \sum_k 2^{-k(n+\ -(n-1))} \chi_{\tilde{\Delta}^k_z}(x) \right. \\ &\quad \left. \times \frac{dz}{\delta^n(z)} k_t(x)\, d\sigma(x) \right] \cdot Q(t). \end{aligned}$$

Proceeding exactly as in Theorem 3.1,

$$\int_{z \in AZ^{(1)}} a(z) \sum_k 2^{-k(\eta + (n-1))} \chi_{\tilde{\Delta}_z^k}(x) \frac{dz}{\delta^n(z)} \leqslant g_{1+\eta/(n-1)}(a)(x),$$

so that

$$\dot{Q}(t) \leqslant CQ(t) + \left(\int_\Delta f(x)\, k_t(x)\, g^{(1)}_{1+\eta/(n-1)}(a)(x)\, d\sigma(x) \right) \cdot Q(t)$$

$$\leqslant C\, \|a\|_\infty\, Q(t) + \|g_{1+\eta/(n-1)}(a)\|_\infty\, Q^2(t),$$

and taking $\delta = \eta/(n-1)$ and $\|g^{(1)}_{1+\delta}(a)\|_\infty < \varepsilon$, where $\varepsilon > 0$ is small enough, this implies that $Q(1) < \infty$, hence that $\omega_1 \in B^\infty(d\sigma)$, as required.

Proof of (ii). Under the assumption that $a(z) < \varepsilon$ for all $z \in B$, if $\varepsilon > 0$ is small enough, then using Lemma 2.2 and its corollary we get both

$$\frac{\omega_t(\Delta_z)}{\omega_t(\tilde{\Delta}_z^k)} \leqslant C \left[\frac{\sigma(\Delta_z)}{\sigma(\tilde{\Delta}_z^k)} \right]^{1-\delta'}$$

and

$$K_t(z, x) \leqslant C \sum_k 2^{-k(1-\delta')} \frac{\chi_{\tilde{\Delta}_z^k}(x)}{\omega_t(\tilde{\Delta}_z^k)},$$

and substituting this into the expression from Theorem 3.1 that controls $|\dot{Q}(t)|$ (where here $Q(t) = B^\infty(d\sigma)$ norm of ω_t, and ω_t is the harmonic measure associated to $L_t = (1-t)L_0 + tL_1$), we have

$$|\dot{Q}(t)| \leqslant CQ(t) + C \int_\Delta f(x)\, k_t(x)\, g^{(1)}_{n/(n-1)-\delta}(a)(x)\, d\sigma(x)$$

$$\left(\delta = \left(\frac{n}{n-1} \right) \delta' \right)$$

$$\leqslant C(1 + \|g^{(1)}_{n/(n-1)-\delta}(a)\|_{L^\infty})\, Q(t).$$

Since we may take δ' (hence δ) arbitrarily small by making ε sufficiently small, part (ii) of Theorem 3.2 is proved.

Now we prove a large perturbation theorem with different applications, where we assume a Carleson condition on the disagreement function $a(z)$, rather than an L^∞ one.

THEOREM 3.3. *Let $L_i u = \operatorname{div}(A_i \nabla u)$ in B, $i = 0, 1$ be elliptic operators, and define $a(z)$ as above. Let $1 < p < \infty$ and assume that the L^p Dirichlet problem is solvable for L_0. Suppose*

$$\left\| \left(\int_{\Gamma^d(x)} a^2(z) \left(\frac{d}{\delta}\right)^{2-\varepsilon} \frac{dz}{\delta^n} \right)^{1/2} \right\|_{L^{p',1}(d\sigma/\sigma(\Delta);\, \Delta)} \leqslant C, \qquad (\sim)$$

for all surface balls $\Delta = \Delta(x_0, d)$. (Here $L^{p',1}$ denotes the Lorentz space, and $\Gamma^d(x)$ is the cone with vertex x truncated at height d, while $\varepsilon > 0$ in $(\sim)$ depends only on the ellipticity constants of the L_i.) Then the L^p Dirichlet problem is also solvable for L_1.

Proof of Theorem 3.3. Following Dahlberg [1] as in Theorem 3.1, we let $L_t = (1-t)L_0 + tL_1$, $t \in [0, 1]$. We let $\Delta_0 = \Delta(x_0; d_0)$ be a surface ball and $A = B(x_0; 2d_0) \cap B$. Assume f is a continuous function supported in Δ_0 such that $\|f\|_{L^p(d\sigma/\sigma(\Delta))} = 1$. Suppose u_t and h_t are defined in A by $L_t u_t = 0$ in A with $u_t|_{\partial A} = f$ while $L_t h_t = \varphi$ in A with $h_t|_{\partial A} = 0$ (φ is a positive bump function of integral one supported in $B((1 - \frac{3}{2} d_0)x_0; d_0/10))$. Further, suppose $\varepsilon(z) = |A_1(z) - A_0(z)|$, ω_t denotes the harmonic measure of L_t in A, and $Q(t)$ denotes the $B^{p'}(d\sigma)$ norm of $d\omega_t = k_t\, d\sigma$. Just as in [1] or Theorem 3.1, it suffices to prove the estimate

$$\int_{A_1} \varepsilon(z)\, |\nabla u_t(z)|\, |\nabla h_t(z)|\, dz \leqslant CQ(t),$$

where $A_1 = A \cap B(x_0; \frac{3}{2} d_0))$. We proceed as follows:

$$\int_{AZ_1} \varepsilon(z)\, |\nabla u_t(z)|\, |\nabla h_t(z)|\, dz$$

$$\leqslant \int_{x \in \Delta(x_0; 2d_0)} \int_{\Gamma^{(3/2)d_0}(x)} \varepsilon(z)\, |\nabla u_t(z)|\, \delta\, |\nabla h_t(z)| \frac{dz}{\delta^n}\, d\sigma(x).$$

Then

$$\int_{\Gamma^{(3/2)d_0}(x)} \varepsilon(z)\, \delta(z)\, |\nabla u_t(z)|\, |\nabla h_t(z)| \frac{dz}{\delta^n(z)}$$

$$\leqslant \left(\int_{\Gamma^{(3/2)d_0}(x)} |\nabla u_t(z)|^2\, \delta^{2-n}(z)\, dz \right)^{1/2} \left(\int_{\Gamma^{(3/2)d_0}(x)} \varepsilon^2(z)\, |\nabla h_t(z)|^2 \frac{dz}{\delta^n} \right)^{1/2}$$

$$\leqslant S(f)(x) \left(\int_{\Gamma^{(3/2)d_0}(x)} \varepsilon^2(z)\, |\nabla h_t(z)|^2 \frac{dz}{\delta^n} \right)^{1/2}.$$

We now estimate

$$\int_{\Gamma^{(3/2)d_0(x)}} \varepsilon^2(z)\,|\nabla h_t(z)|^2 \frac{dz}{\delta_n}$$

$$\leqslant C \int_{z \in \Gamma^{(3/2)d_0(x)}} a^2(z) \int_{B(z;\,\delta(z)/10)} |\nabla h_t(y)|^2 \frac{dy}{\delta^n(z)} \frac{dz}{\delta^n(z)}.$$

Cacciopolli's inequality implies that this last quantity is majorized by

$$C \int_{z \in \Gamma^{(3/2)d_0(x)}} a^2(z) \int_{B(z;\,\delta(z)/2)} h_t^2(y)\,dy \frac{dz}{\delta^{2n+2}(z)}$$

$$\leqslant C \int_{\Gamma^{2d_0(x)}} a^2(z)\,h_t^2(z) \frac{dz}{\delta^{n+2}}.$$

We have, by standard estimates on the Green's function,

$$h_t(z) \leqslant C d_0^{2-n} \left(\frac{\delta(z)}{d_0}\right)^{\varepsilon/2},$$

for some $\varepsilon > 0$ depending only on the ellipticity constant of L_t. Thus

$$\int_{\Gamma^{2d_0(x)}} a^2(z)\,h_t^2(z) \frac{dz}{\delta^{n+2}} \leqslant \int_{\Gamma^{2d_0(x)}} a^2(z)\,d_0^{4-2n} \left(\frac{\delta}{d_0}\right)^{\varepsilon} \frac{dz}{\delta^{n+2}}$$

$$\leqslant d_0^{2-2n} \int_{\Gamma^{2d_0(x)}} a^2(z) \left(\frac{d_0}{\delta}\right)^{2-\varepsilon} \frac{dz}{\delta^n}.$$

This gives

$$\int_{A_1} \varepsilon(z)\,|\nabla u_t(z)|\,|\nabla h_t(z)|\,dz$$

$$\leqslant \int_{\Delta_0} S(f)(x) \left[\int_{\Gamma^{2d_0(x)}} a^2(z) \left(\frac{d_0}{\delta}\right)^{2-\varepsilon} \frac{dz}{\delta^n}\right]^{1/2} \frac{d\sigma}{\sigma(\Delta_0)}$$

$$\leqslant C\,\|S(f)\|_{L^{p,\infty}(d\sigma/\sigma(\Delta_0);\,\Delta_0)}$$

$$\times \left\| \left(\int_{\Gamma^{2d_0(x)}} a^2(z) \left(\frac{d_0}{\delta}\right)^{2-\varepsilon} \frac{dz}{\delta^n}\right)^{1/2} \right\|_{L^{p',1}(d\sigma/\sigma(\Delta_0);\,\Delta_0)}$$

$$\leqslant C\,\|S(f)\|_{L^{p,\infty}(d\sigma/\sigma(\Delta_0);\,\Delta_0)}.$$

We now estimate $\|S(f)\|_{L^{p,\infty}(d\sigma/\sigma(\Delta_0);\partial A)}$. First, we observe that for $x \in \partial A - \Delta(x_0; \frac{3}{2} d_0)$, $S(f)(x) \leqslant C \int_{\Delta_0} f\, d\omega_t \leqslant CQ(t)$. This is easily seen by the Cacciopolli estimate

$$S^2(f)(x) = \int_{\Gamma^{(3/2)d_0}(x)} |\nabla u_t|^2 \delta^2 \frac{dz}{\delta^n} \leqslant C \int_{\Gamma^{(3/2)d_0}(x)} u_t^2(z) \frac{dz}{\delta^n},$$

and the standard estimate for the kernel function showing that

$$u_t(z) \leqslant C \int_{\Delta_0} f\, d\omega_t \cdot 2^{-j\beta},$$

whenever $z \in \Gamma^{(3/2)d_0}(x)$ and $\delta(z) \leqslant 2^{-j} |x - x_0|$ ($\beta > 0$ depends only on the ellipticity of L_t).

Now, setting $\tilde{\sigma} = \sigma/\sigma(\Delta_0)$, we see that

$$\begin{aligned} \|Sf\|_{L^{p,\infty}(d\tilde{\sigma};\partial A)} &= \sup_{\alpha > 0} \alpha[\tilde{\sigma}\{Sf > \alpha\}]^{1/p} \\ &\leqslant CQ(t) + \sup_{\alpha > CQ(t)} \alpha \cdot [\tilde{\sigma}\{Sf > \alpha\}]^{1/p}. \end{aligned}$$

In the second term above, note that when $\alpha > CQ(t)$, $\{Sf > \alpha\} \subseteq \Delta(x_0; \frac{3}{2} d_0)$. Also, we have the good λ inequality

$$\omega_t\{Sf > 2\alpha, M_{\omega_t} f < \gamma\alpha\} \leqslant C\gamma^\delta \omega_t\{Sf > \alpha\}.$$

It is easily seen (see Lemmas 4.1 and 4.2 below) that the assumptions we are making on $a(z)$ imply that $a^2\, dz/\delta$ is a Carleson measure. Therefore, according to the Theorem on Large Perturbations [2], it follows that $\sigma \in A^\infty(d\omega_t)$ with an A^∞ norm independent of t. We have, for $\alpha > CQ(t)$,

$$\tilde{\sigma}\{Sf > 2\alpha\} \leqslant C\gamma^{\delta'} \tilde{\sigma}\{Sf > \alpha\} + \tilde{\sigma}\{M_{\omega_t}(f) > \alpha\},$$

and it follows that

$$\|Sf\|_{L^{p,\infty}(d\tilde{\sigma};\partial A)} \leqslant CQ(t) + \|M_{\omega_t}(f)\|_{L^{p,\infty}(d\tilde{\sigma})}.$$

Recall that $M_{\omega_t}(f)(xc) \leqslant Q(t)\, M_\sigma(f^p)^{1/p}$, so that

$$\|M_{\omega_t}(f)\|_{L^{p,\infty}(d\tilde{\sigma})} \leqslant CQ(t)$$

and it follows that

$$\|S(f)\|_{L^{p,\infty}(d\tilde{\sigma};\partial A)} \leqslant CQ(t).$$

This shows that

$$\int_{A_1} \varepsilon(z)\, |\nabla u_t(z)|\, |\nabla h_t(z)|\, dz \leqslant CQ(t)$$

and completes the proof of Theorem 3.3.

4. Extensions of the Perturbation Theorems by Interpolation

Let L_0 and L_1 be elliptic operators in B, $L_i u = \operatorname{div}(A_i \nabla u)$ and let $a(z) = \operatorname{ess\,sup}_{y \in B(z;\, \delta(z)/2)} |A_0(y) - A_1(y)|$. Suppose $1 < p < \infty$, and that we know that the $L^{p'}$ Dirichlet problem is solvable for L_0. Then, at this point we know that the condition

$$g^{(1)}_{(n-2)/(n-1)+\varepsilon}(a) \in L^\infty$$

or

$$\left\| \left(\int_{\Gamma^d(x)} \left[a(z) \left(\frac{d}{\delta}\right)^{1-\varepsilon} \right]^2 \frac{dz}{\delta^n} \right)^{1/2} \right\|_{L^{p,1}(df\sigma/\sigma(\Delta_0);\, \Delta_0)} \leqslant C$$

(ε depending only on the ellipticity of the L_i) implies that the $L^{p'}$ Dirichlet problem is solvable for L_1. In this section, we improve on this by showing that in dimension $n \geqslant 3$ either of the conditions

$$g^2_\lambda(a)(x) = \int_B a^2(z) \left(\frac{\delta(z)}{|x-z|} \right)^{n-2} \frac{dz}{\delta^n} \in L^\infty$$

or

$$\frac{1}{\sigma(\Delta(x;d))} \int_{S(\Delta(x;d))} a^2(z) \left(\frac{d}{\delta}\right)^p \frac{dz}{\delta}$$

$$\leqslant C, \qquad \text{for all surface balls } \Delta(x;d),$$

implies that L_1 has a solvable $L^{p'}$ Dirichlet problem. To do this, we require several lemmas which we now prove. In our discussion below we always assume that $L_i u = \operatorname{div}(A_i \nabla u)$ are elliptic in B and that the $L^{p'}$ Dirichlet problem is solvable for L_0. Then we have:

Lemma 4.1. *Suppose $\varepsilon > 0$ (depending only on the ellipticity of the L_i) is such that the condition*

$$\left\| \left(\int_{\Gamma^d(x)} \left[a(z) \left(\frac{d}{\delta}\right)^{1-\varepsilon} \right]^2 \frac{dz}{\delta^n} \right)^{1/2} \right\|_{L^{p,1}(d\sigma/\sigma(\Delta_0);\, \Delta_0)} \leqslant C$$

for all surface balls $\Delta = \Delta(x_0; d)$ implies that the $L^{p'}$ Dirichlet problem is solvable for L_1. Suppose $\tilde{p}$ satisfies $\tilde{p} > p$, but $(1-\varepsilon/2)\tilde{p} < p$. Then the condition

$$\frac{1}{\sigma(\Delta)}\int_{S(\Delta)} a(z)^{\tilde{p}}\left(\frac{d}{\delta}\right)^p \frac{dz}{\delta} \leqslant C$$

for all surface balls $\Delta = \Delta(x_0; d)$ also implies that the $L^{p'}$ Dirichlet problem is solvable for L_1.

Proof. Since $\|f\|_{L^{\tilde{p}}(d\sigma/\sigma(\Delta_0))} \leqslant c\,\|f\|_{L^{p,1}(d\sigma/\sigma(\Delta_0))}$ for all functions f, it follows that

$$\left\|\left(\int_{\Gamma^d(x)}\left[a(z)\left(\frac{d}{\delta}\right)^{1-\varepsilon}\right]^2 \frac{dz}{\delta^n}\right)^{1/2}\right\|_{L^{\tilde{p}}(d\sigma/\sigma(\Delta_0))} \leqslant C$$

for all Δ implies that the $L^{p'}$ Dirichlet problem is solvable for L_1.

Now we may assume that the function $a(z) = \operatorname{ess\,sup}_{y\in B(z;\,\delta(z)/100)} |A_0(y) - A_1(y)|$ and letting $\tilde{a}(z) = \operatorname{ess\,sup}_{y\in B(z;\,\delta(z)/2)} |A_0(y) - A_1(y)|$ it is clear that for $\Gamma_k^d(x) = \Gamma^d(x) \cap \{z \mid 2^{-k}d < \delta(z) \leqslant 2\cdot 2^{-k}d\}$ we have

$$\left(\int_{\Gamma^d(x)}\left[a\left(\frac{d}{\delta}\right)^{1-\varepsilon}\right]^2 \frac{dz}{\delta^n}\right)^{1/2} \leqslant \left(\sum\int_{\Gamma_k^d(x)}\left[\alpha_k\left(\frac{d}{\delta}\right)^{1-\varepsilon}\right]^2 \frac{dz}{\delta^n}\right)^{1/2},$$

where $\alpha_k = \inf_{z\in\Gamma_k^d(x)} \tilde{a}(z)$. This is, in turn, dominated by

$$C\left(\sum\int_{\Gamma_k^d(x)}\left[\alpha_k\left(\frac{d}{\delta}\right)^{1-\varepsilon/2}\right]^{\tilde{p}} \frac{dz}{\delta^n}\right)^{1/\tilde{p}}$$
$$\leqslant C\left(\int_{\Gamma^d(x)}\left[\tilde{a}(z)\left(\frac{d}{\delta}\right)^{1-\varepsilon/2}\right]^{\tilde{p}} \frac{dz}{\delta^n}\right)^{1/\tilde{p}}.$$

Therefore, we see that

$$\left\|\left(\int_{\Gamma^d(x)}\left[a\left(\frac{d}{\delta}\right)^{1-\varepsilon}\right]^2 \frac{dz}{\delta^n}\right)^{1/2}\right\|^{\tilde{p}}_{L^{\tilde{p}}(d\sigma/\sigma(\Delta))}$$
$$\leqslant C\left\|\left(\int_{\Gamma^d(x)}\left[a\left(\frac{d}{\delta}\right)^{1-\varepsilon/2}\right]^{\tilde{p}} \frac{dz}{\delta^n}\right)^{1/\tilde{p}}\right\|^{\tilde{p}}_{L^{\tilde{p}}(d\sigma/\sigma(\Delta))}$$
$$\leqslant \frac{C}{\sigma(\Delta)}\int_{S(\Delta)} \tilde{a}^{\tilde{p}}(z)\left(\frac{d}{\delta}\right)^{(1-\varepsilon/2)\tilde{p}} \frac{dz}{\delta}$$
$$\leqslant \frac{C}{\sigma(\Delta)}\int_{S(\Delta)} \tilde{a}(z)^{\tilde{p}}\left(\frac{d}{\delta}\right)^p \frac{dz}{\delta},$$

so that if this last quantity is bounded uniformly for all Δ, it follows that the same is true for

$$\left\| \left(\int_{\Gamma^d(x)} \left[a \left(\frac{d}{\delta} \right)^{1-\varepsilon} \right]^2 \frac{dz}{\delta^n} \right)^{1/2} \right\|_{L^{\tilde{p}}(d\sigma/\sigma(\Delta))},$$

hence $\omega_{L_1} \in B^p(d\sigma)$ holds. This proves Lemma 4.1.

We also need the following:

LEMMA 4.2. *If* $1/\sigma(\Delta) \int_{S(\Delta)} a^q(z)(d/\delta)^{\varepsilon} (dz/\delta) \leqslant C$ *for some* $\varepsilon > 0$, *some* q *with* $1 < q < \infty$, *and for all surface balls* $\Delta = \Delta(x, d)$, *it follows that* $1/\sigma(\Delta) \int_{S_\eta(\Delta)} a^2(z)\, dz/\delta \to 0$ *uniformly in* Δ *as* $n \to 0$, *where* $S_\eta(\Delta) = S(\Delta) \cap \{z \in B \mid \delta(z) < \eta \operatorname{diam}(\Delta)\}$.

Proof. Define $S^k(\Delta) = S(\Delta) \cap \{z \in B \mid 2^{-k}d < \delta(z) < 2 \cdot 2^{-k}d\}$. Then from our hypothesis, it follows that

$$\frac{1}{\sigma(\Delta)} \int_{S^k(\Delta)} a^q(z) \frac{dz}{\delta} \leqslant C 2^{-k\varepsilon}.$$

If $q \geqslant 2$, then

$$\frac{1}{\sigma(\Delta)} \int_{S^k(\Delta)} a^2(z) \frac{dz}{\delta} \leqslant C \left(\frac{1}{\sigma(\Delta)} \int_{S^k(\Delta)} a^q(z) \frac{dz}{\delta} \right)^{2/q} \leqslant C 2^{-k(2/q)\varepsilon}.$$

if $q < 2$, then

$$\frac{1}{\sigma(\Delta)} \int_{S^k(\Delta)} a^2(z) \frac{dz}{\delta} \leqslant \|a\|_{L^\infty}^{2-q} \frac{1}{\sigma(\Delta)} \int_{S^k(\Delta)} a^q(z) \frac{dz}{\delta} \leqslant C' 2^{-k\varepsilon}.$$

In either case, summing on k finishes the proof.

Now we are ready to prove the main theorems of this section, which extend the perturbation results of the previous section to a more familiar setting.

THEOREM 4.3. *Suppose that* $L_i u = \operatorname{div}(A_i \nabla u)$, $i = 0, 1$, *are elliptic, and that the* $L^{p'}$ *Dirichlet problem is solvable for* L_0. *Define,*

$$a(z) = \operatorname*{ess\,sup}_{y \in B(z;\, \delta(z)/2)} |A_0(y) - A_1(y)|,$$

and suppose that $(1/\sigma(\Delta)) \int_{S(\Delta)} a^2(z)(d/\delta)^p (dz/\delta) \leqslant C$ *for all surface balls* $\Delta = \Delta(x; d)$. *Then the* $L^{p'}$ *Dirichlet problem is also solvable for* L_1.

Proof. We know that there is an exponent $\tilde{p}$ such that if $1/\sigma(\Delta)\int_{S(\Delta)} a^{\tilde{p}}(z)(d/\delta)^p\,(dz/\delta)\leqslant C$ for all surface balls Δ, then $\omega_{L_1}\in B^p(d\sigma)$. In the case $2\leqslant\tilde{p}$ our theorem is clear, since then

$$\frac{1}{\sigma(\Delta)}\int_{S(\Delta)} a^{\tilde{p}}(z)\left(\frac{d}{\delta}\right)^p\frac{dz}{\delta}\leqslant\|a\|_\infty^{\tilde{p}-2}\frac{1}{\sigma(\Delta)}\int_{S(\Delta)} a^2(z)\left(\frac{d}{\delta}\right)^p\frac{dz}{\delta}\leqslant C'.$$

In case $2>\tilde{p}$, we argue by defining an intermediate perturbation M of L_0 given by

$$Mu=\operatorname{div}(A\,\nabla u),$$

where

$$A(z)=\begin{cases}A_1(z) & \text{if } |A_0(z)-A_1(z)|\leqslant\zeta\\ A_0(z) & \text{if } |A_0(z)-A_1(z)|>\zeta.\end{cases}$$

Set

$$a_1(z)=\operatorname*{ess\,sup}_{y\in B(z;\,\delta(z)/2)}|A(y)-A_0(y)|$$

and

$$a_2(z)=\operatorname*{ess\,sup}_{y\in B(z;\,\delta(z)/2)}|A_1(y)-A(y)|.$$

Then we claim that the first perturbation $L_0\to M$ (the "small part" of $L_0\to L_1$) satisfies the hypothesis of Dahlberg's Theorem on Small Perturbations provided $\zeta>0$ is sufficiently small. In fact, for $0<\eta<1$ we have

$$\frac{1}{\sigma(\Delta)}\int_{S(\Delta)} a_1^2\frac{dz}{\delta}=\frac{1}{\sigma(\Delta)}\int_{S_\eta(\Delta)} a^2\frac{dz}{\delta}+\left(\frac{1}{\sigma(\Delta)}\int_{S\setminus S_\eta(\Delta)}\frac{dz}{\delta}\right)\cdot\zeta^2.$$

By Lemma 4.2, $(1/\sigma(\Delta))\int_{S_\eta(\Delta)} a^2(dz/\delta)\to 0$ uniformly in Δ as $\eta\to 0$. So we choose η to make this small, and then observe that the second term is $O(\zeta^2\log(1/\eta))$ so that we can also make this term arbitrary small by choosing a $\zeta>0$ small enough. This means that by a correct choice of ζ, the Carleson norm of $a_1^2\,dz/\delta$ is small enough to imply that $\omega_M\in B^p(d\sigma)$. As for the "large part" of the perturbation $L_0\to L_1$, namely $M\to L_1$, we observe that $a_2(z)=0$ or $a_2(z)\geqslant\zeta$ for all $z\in B$. Therefore,

$$\frac{1}{\sigma(\Delta)}\int_{S(\Delta)} a^{\tilde{p}}(z)\left(\frac{d}{\delta}\right)^p\frac{dz}{\delta}\leqslant\zeta^{\tilde{p}-2}\frac{1}{\sigma(\Delta)}\int_{S(\Delta)} a^2(z)\left(\frac{d}{\delta}\right)^p\frac{dz}{\delta}\leqslant C\zeta^{\tilde{p}-2}$$

and if follows that $\omega_{L_1}\in B^p(d\sigma)$, as well. This proves Theorem 4.3.

We also prove a theorem analogous to the preceding one, only extending an L^∞ condition rather than a Carleson condition.

THEOREM 4.4. *Let* $L_i u = \operatorname{div}(A_i \nabla u)$, $i = 0, 1$, *be elliptic operators in* B, *and* $a(z) = \operatorname{ess\,sup}_{y \in B(z;\, \delta(z)/2)} |A_0(y) - A_1(y)|$. *Suppose the* L^p *Dirichlet problem is solvable for* L_0. *Then if*

$$g_\lambda(a)(x) = \left(\int_B a^2(z) \left(\frac{\delta(z)}{|z-x|} \right)^{n-2} \frac{dz}{\delta^n(z)} \right)^{1/2} \in L^\infty(S^{n-1}),$$

then the L^p *Dirichlet problem is also solvable for* L_1.

Proof. Note that if $\Delta = \Delta(x_0; d)$ is a surface ball, then our hypothesis implies that

$$C \geqslant \frac{1}{\sigma(\Delta)} \int_\Delta \int_{\Gamma^d(x)} a^2(z) \left(\frac{\delta(z)}{|z-x|} \right)^{n-2} \frac{dz}{\delta^n(z)}\, d\sigma(x)$$

$$\geqslant c' \frac{1}{\sigma(\Delta)} \int_{S(\Delta)} a^2(z) \left(\frac{d}{\delta} \right) \frac{dz}{\delta},$$

hence it follows that $1/\sigma(\Delta) \int_{S_\eta(\Delta)} a^2\, dz/\delta \to 0$ uniformly in Δ as $\eta \to 0$. Thus, if we define M exactly as in the preceding theorem, a repetition of the proof given there also proves Theorem 4.4 (only here we apply the Theorem 3.1 on $g_\lambda^{(1)}(a)$ to handle the "large part").

Before proceeding with the applications of these results, we should remark that Theorems 4.3 and 4.4 are not comparable; i.e., they apply in different situations. For example, in the case of operators whose coefficients are homogeneous of degree zero (with respect to the usual dilations), unless certain additional assumptions are made, only the Carleson measure formulation (Theorem 4.3) applies. However, the reader should note that the condition $g_{(n-2)/(n-1)}(a) \in L^\infty$ is closely related to (and slightly stronger than) the condition $1/\sigma(\Delta) \int_{S(\Delta)} a^2(d/\delta)\, dz/\delta \leqslant C$. We observe that the power of (d/δ) here is only one and this is to be contrasted to Theorem 4.3 which requires the power of (d/δ) to be p for perturbing operators whose harmonic measures are in B^p. So in this sense the Carleson condition represents a much stronger assumption than the L^∞ one assumed in Theorem 4.4.

Another remark that should be made is that in the dimension $n = 2$ versions of Theorems 4.3 and 4.4, they involve the ellipticity constants of the operators. For instance, when $n = 2$, for Theorem 4.4, we have the following:

If $n=2$ and the L^p Dirichlet problem is solvable for L_0, then there exists $\varepsilon>0$ depending only on the ellipticity constants of L_0 and L_1 so that the condition

$$\int_B a^2(z)\left(\frac{\delta(z)}{|z-x|}\right)^{\varepsilon}\frac{dz}{\delta^n}\in L^\infty$$

implies that the L^p Dirichlet problem is also solvable for L_1.

Next we should point out that it is easy to see that the method of proof of Theorems 4.3 and 4.4 actually leads to somewhat stronger versions of these results. These are as follows:

THEOREM 4.3′. *Suppose L_i are elliptic in B and define $a(z), z\in B$, as above. Let $\omega\colon[0,\infty)\to\mathbb{R}$ satisfy $\omega(0)=0$, where ω is strictly increasing, continuous, and convex in a neighborhood of 0. Suppose that $\int_0^1\omega^{-1}(\delta)\,d\delta/\delta<\infty$. Then either of the conditions*

$$\int_B \omega(a^2)\left(\frac{\delta(z)}{|z-x|}\right)^{n-2}\frac{dz}{\delta^n}\in L^\infty$$

or

$$\frac{1}{\sigma(\Delta)}\int_{S(\Delta)}\omega(a^2)\left(\frac{d}{\delta}\right)^p\frac{dz}{\delta}\leqslant C\qquad \forall\Delta=\Delta(x;d)\subseteq\partial B,$$

together with the solvability of the $L^{p'}$ Dirichlet problem for L_0 implies the solvability of the $L^{p'}$ Dirichlet problem for L_1.

The proof of Theorem 4.3′ is essentially the same as that of Theorems 4.3 and 4.4, except that one needs to apply a strengthened form of Lemma 4.2:

LEMMA 4.2′. *Suppose ω is as in Theorem 4.3′, and is convex on $[0,\delta_0]$. Suppose that $a^2(z)\leqslant\delta_0$, for all $z\in B$, and satisfies*

$$\frac{1}{\sigma(\Delta)}\int_{S(\Delta)}\omega(a^2)\left(\frac{d}{\delta}\right)\frac{dz}{\delta}\leqslant C\qquad \forall\Delta=\Delta(x;d)\subseteq\partial B.$$

Then

$$\frac{1}{\sigma(\Delta)}\int_{S^\eta(\Delta)}a^2\frac{dz}{\delta}\to 0$$

uniformly in Δ as $\eta\to 0$ where $S^\eta(\delta=\{z\in B\mid\delta(z)<\eta\operatorname{diam}(\Delta)\}\cap S(\Delta)$.

Proof of Lemma 4.2′. Let $S_k(\Delta)=\{z\in B \mid 2^{-k}d<\delta(z)\leqslant 2^{-2(k-1)}d\}\cap S(\Delta)$. Then by assumption

$$\frac{1}{\sigma(\Delta)}\int_{S_k(\Delta)}\omega(a^2)\frac{dz}{\delta}\leqslant C\cdot 2^{-k},$$

and applying Jensen's inequality gives

$$\frac{1}{\sigma(\Delta)}\int_{S_k(\Delta)}a^2\frac{dz}{\delta}\leqslant \omega^{-1}(C\cdot 2^{-k}),$$

and the lemma follows at once from the convergence of $\sum_k \omega^{-1}(C\cdot 2^{-k})$.

Finally, we remark that when certain additional assumptions are made on the operators L_0 and L_1, we can derive extensions of our theorems above. For instance, in the special case where we are perturbing the Laplacian, we can obtain, using the methods above, a large perturbation version of the Fabes–Jerison–Kenig Theorem [4] mentioned in Section 1. As is easily seen by use of the Comparison Theorem, the solvability of the L^p Dirichlet problem is not affected by changing the coefficients of an elliptic operator in a proper subball $B(0;r)\subseteq B$ with $0<r<1$. We now make some obvious observations which relate, for perturbations of the Laplacian, the Fabes–Jerison–Kenig Theorem to our theorems here.

Let $Lu=\operatorname{div}(A\,\nabla u)$ be an elliptic operator with coefficients continuous in $\bar{B}$ and such that $A(z)\equiv I$ for all $z\in\partial B$. Suppose, as in the Fabes–Jerison–Kenig Theorem, that

$$\int_0^1 \omega^2(\delta)\frac{d\delta}{\delta}<\infty, \qquad (**)$$

where $\omega(\delta)$ denotes the uniform modulus of continuity at the boundary of B. Define, as usual, $a(z)=\operatorname{ess\,sup}_{y\in B(z;\,\delta(z)/2)}|A(y)-I|$, and let $\varepsilon>0$, $\gamma>0$, and $\lambda>1$. Then, by setting $A(z)\equiv I$ for all $z\in B(0;r)$ and choosing $0<r<1$ sufficiently close to 1 (depending on $\varepsilon>0$, $\gamma>0$, and $\lambda>1$), we easily see that $(**)$ implies

(i) $a(z)<\varepsilon$, for all $z\in B$,

(ii) $\|g_\lambda(a)\|_{L^\infty}<\varepsilon$, and

(iii) $\|S_\gamma(a)\|_{L^\infty}<1$

(S_γ is the Lusin area functional taken with respect to a cone of aperture γ: $S_\gamma(a)(x)=(\int_{\Gamma_\gamma(x)}a^2(z)\,dz/\delta^n)^{1/2}$). Our extension of the theorem in [4] for perturbations of the Laplacian can be stated as follows:

Choose and fix $1<\lambda<n/(n-1)$. Let ε be given so small that $\|g_\lambda^{(1)}(a)\|_\infty<\varepsilon$ implies preservation of the solvability of the L^p Dirichlet problem for all p. Let L_0 and L_1 be elliptic operators with bounded measurable coefficients in B: $L_i u=\operatorname{div}(A_i\nabla u)$ for $i=0, 1$. Assume $1<p<\infty$ is fixed and that the L^p Dirichlet problem is solvable for L_0. Define $a(z)=\operatorname{ess\,sup}_{y\in B(z;\,\delta(z)/2)}|A_0(y)-A_1(y)|$.

THEOREM 4.5. *Suppose that $|A_i(z)-I|<\varepsilon$ for all $z\in B$ and $i=0, 1$. Assume that $g_\lambda(a)\in L^\infty$ and that $\|S_\gamma(a)\|_{L^\infty}=M$. Then if γ is large enough (the largeness of γ depends on p, $\|\omega_{L_0}\|_{B^{p'}}$, and M), it follows that the L^p Dirichlet problem is solvable for L_1.*

Proof. The proof of Theorem 4.5 is almost identical to that of Theorem 4.4. We split the perturbation from L_0 to L_1 into two perturbations: L_0 to M and M to L_1. Just as in Theorem 4.4, we show that $a_1^2\,dz/\delta$ is a Carleson measure of small norm. This uses the assumption $S_\gamma(a)\leqslant M$ for γ large enough instead of $g_{(n-2)/(n-1)}(a)\in L^\infty$ which we used in 4.4. This shows that $\omega_M\in B^{p'}(d\sigma)$. The second perturbation, M to L_1, satisfies $g_\lambda^{(1)}(a)\in L^\infty$ and the choice of ε guarantees that $\omega_{L_1}\in B^{p'}(d\sigma)$, finishing the proof of Theorem 4.5.

5. SOME EXAMPLES AND APPLICATIONS

In this section, we give two examples to illustrate concretely where our results apply. The first result concerns "regions of arbitrary perturbation." By this we mean the following: We are given a region $\mathscr{R}$ and an open subset $\mathscr{O}\subseteq\mathscr{R}$, and an elliptic operator L_0 in $\mathscr{R}$ whose L^p Dirichlet problem is solvable (for a fixed p). Suppose that we now change the operator L_0 by redefining the value of its coefficients on the set $\mathscr{O}$ (keeping them unchanged outside $\mathscr{O}$) to get a new operator L_1. Suppose that no matter what value the coefficients of L_1 have on $\mathscr{O}$, we have that the L^p Dirichlet problem for L_1 is solvable. Then we say that $\mathscr{O}$ is a region of arbitrary perturbation (this may depend on p). Our first result states that a cone is a region of arbitrary perturbation for certain values of p depending on the dimension. We show that this follows immediately from Theorem 3.3. We also give a direct elementary proof of this fact. Then as a second application, we study operators whose coefficients are homogeneous, and show that in many cases those whose coefficients have parabolic homogeneity have better properties than those whose coefficients have the usual isotropic homogeneity.

THEOREM 5.1. *Let L_0 and L_1 be elliptic operators in $B_+ = B \cap \mathbb{R}^n_+$. Suppose that the L^p Dirichlet problem is solvable for L_0 and that $n \geqslant 3$, $p \geqslant (n-1)/(n-2)$. Then if the coefficients of L_0 and L_1 are identical outside the cone $\Gamma^{1/2}(0)$, the L^p Dirichlet problem is also solvable for L_1.*

First proof. Letting $L_i u = \operatorname{div}(A_i \nabla u)$, $i = 0, 1$, and defining $a(z)$ as usual, we easily verify that for $|x| \leqslant d$ we have

$$\left(\int_{\Gamma^d(x)} a^2(z) \left(\frac{d}{\delta}\right)^{2-\varepsilon} \frac{dz}{\delta^n} \right)^{1/2} \leqslant \frac{C}{|x/d|^{1-\varepsilon/2}},$$

Then Theorem 5.1 follows immediately from Theorem 3.3 one we observe that

$$\frac{1}{d^{n-1}} \int_{|x|<d} \left(\frac{1}{|x/d|^{1-\varepsilon/2}} \right)^{p'} dx$$

is bounded independent of d, provided $p' < (n-1)/(1-\varepsilon/2)$.

Second proof. Let Δ be a surface ball, which may easily be assumed to be centered at 0, say $\Delta = \Delta(0; d)$. Set $A_k = \{x \in \Delta \mid 2^{-k} d \leqslant |x| < 2 \cdot 2^{-k} d\}$ for $k \geqslant 1$. Cover A_k with surface balls $\Delta^k_j = \Delta(x^k_j, d^k_j)$, $1 \leqslant j \leqslant N_n$, where N_n is independent of k, and where Δ^k_j are chosen to make $B(x^k_j; 2d^k_j) \cap \Gamma(0) = \varnothing$. Applying the Comparison Theorem, we see that for $\omega_{L_i} = k_i \, d\sigma$, $i = 0, 1$, we have

$$\frac{(1/\sigma(\Delta^k_j) \int_{\Delta^k_j} k_1^p \, d\sigma)^{1/p}}{1/\sigma(\Delta^k_j) \int_{\Delta^k_j} k_1 \, d\sigma} \leqslant C \frac{(1/\sigma(\Delta^k_j) \int_{\Delta^k_j} k_0^p \, d\sigma)^{1/p}}{1/\sigma(\Delta^k_j) \int_{\Delta^k_j} k_0 \, d\sigma} \leqslant C \, \|k_0\|_{B^p},$$

where C is independent of j and j. This can be rewritten as

$$\int_{\Delta^k_j} k_1^p \, d\sigma \leqslant C^p \, \|k_0\|^p_{B^p} \cdot [\omega_{L_1}(\Delta^k_j)]^p \, \sigma(\Delta^k_j)^{1-p}$$

and summing on k and j we have

$$\frac{1}{\sigma(\Delta)} \int_\Delta k_1^p \, d\sigma \leqslant C^p \, \|k_0\|^p_{B^p} \cdot \left[\frac{\omega_{L_1}(\Delta)}{\sigma(\Delta)}\right]^p \sum_{k,j} \left[\frac{\omega_{L_1}(\Delta^k_j)}{\omega_{L_1}(\Delta)}\right]^p \left[\frac{\sigma(\Delta)}{\sigma(\Delta^k_j)}\right]^{p-1}.$$

The proof is then completed by observing that for some $\varepsilon > 0$ depending only on the ellipticity constants of the L_i,

$$\left[\frac{\omega_{L_1}(\Delta^k_j)}{\omega_{L_1}(\Delta)}\right] \leqslant C \left[\frac{\sigma(\Delta^k_j)}{\sigma(\Delta)}\right]^{(n-2)/(n-1)+\varepsilon},$$

so that

$$\sum_{k,j}\left[\frac{\omega_{L_1}(\Delta_j^k)}{\omega_{L_1}(\Delta)}\right]^p\left[\frac{\sigma(\Delta)}{\sigma(\Delta_j^k)}\right]^{p-1}$$

converges provided $(n-2)/(n-1)+\varepsilon-(1-1/p)>0$.

Remark. The same argument as given above in the "first proof" shows that for various sets E of lower dimensional Hausdorff measure, the sawtooth region $\bigcup_{x\in E}\Gamma^{1/2}(x)$ is a region of arbitrary perturbation for the solvability of the L^p Dirichlet problem, where p depends on n and the Hausdorff dimension of E. In this case, checking that our theorems apply involves an analysis of the $n-1$ dimensional Lebesgue measure of the set of points whose distance to E is less than t as $t\to 0$ (just as in Minkowski's definition of dimension). We shall not pursue the details here, however.

Now, let us contrast the properties of elliptic operators whose coefficients are homogeneous with respect to isotropic dilations with those whose coefficients have parabolic homogeneity.

To be precise, let us fix some notation and definitions. We work in $\mathbb{R}^2_+=\{(x,t)\mid x\in\mathbb{R}, t>0\}$. We consider a smooth bounded domain $D\subseteq\mathbb{R}^2_+$ which contains the half-disk $B_+=\{(x,t)\in\mathbb{R}^2_+\mid x^2+t^2<1\}$. We consider the dilations δ_α: $(x,t)\to(\delta x,\delta^\alpha t)$ for some fixed $\alpha>1$, and $\delta>0$, and call these parabolic dilations of order α. Then we have the following result:

THEOREM 5.2. *There is a (real, symmetric, elliptic) C^∞ matrix-valued function on the unit circle S^1 with the following two properties.*

(i) *If this function is extended to be homogeneous of degree zero with respect to the usual dilations, and its extension is denoted $A(x,t)$, then the operator* $\operatorname{div}(A\nabla)$ *in the domain D has its L^p Dirichlet problem solvable only in some range of p: $p>p_0>1$.*

(ii) *If this function is extended to be homogeneous of degree zero with respect to parabolic dilations of any order $\alpha>1$, and we call this homogeneous extension $A_\alpha(x,t)$, then the operator* $\operatorname{div}(A_\alpha\nabla)$ *has its L^p Dirichlet problem solvable in the range $1\leqslant p<\infty$.*

Thus, though it might seem to be intuitively clear that as $\alpha\to 1$ the range of L^p spaces for which the L^p Dirichlet problem is solvable for $\operatorname{div}(A_\alpha\nabla)$ should "converge" to the range of p for which the L^p Dirichlet problem is solvable for $\operatorname{div}(A\nabla)=\operatorname{div}(A_1\nabla)$, this is not the case. As stated in the Introduction, this is a consequence of the theorems in Sections 2 and 3 of this article. To see this we require a lemma:

LEMMA 5.3. *Let the region S_k, $k=1, 2, 3, \dots$, be defined by*

$$S_k = \left\{ (x, t) \in B_+ \;\middle|\; \frac{C}{2^{k+1}} |x|^\alpha < t \leqslant \frac{C}{2^k} |x|^\alpha \right\},$$

where $C > 1$. Let $a(z)$ be a non-negative bounded function supported in S_k. Then for all $\bar{x}$ such that $|\bar{x}| < \frac{1}{2}$, $g_\lambda^{(1)}(a)(\bar{x}, 0) \leqslant M \|a\|_\infty$ where $\lambda > 1$ and M depends on α, λ, and C.

Proof. Define

$$S_k^{(1)} = \{(y, t) \mid |y| > 2|\bar{x}|\} \cap S_k$$

$$S_k^{(2)} = \left\{ (y, t) \;\middle|\; |y| \leqslant 2|\bar{x}|, |\bar{x} - y| > \frac{|\bar{x}|}{2} \right\} \cap S_k$$

and

$$S_k^{(3)} = \left\{ (y, t) \;\middle|\; |y - \bar{x}| \leqslant \frac{|\bar{x}|}{2} \right\} \cap S_k.$$

Then

$$g_\lambda^{(1)}(a)(\bar{x}, 0) \leqslant C' \|a\|_\infty \sum_{i=1}^{3} \int_{S_k^{(i)}} \left(\frac{t}{|\bar{x} - y| + t} \right)^{\lambda(n-1)} \frac{dy\, dt}{t^n}$$

First, we have

$$\begin{aligned} \int_{S_k^{(1)}} \left(\frac{t}{|\bar{x} - y| + t} \right)^{\lambda} \frac{dy\, dt}{t} &\leqslant C' \int_{\substack{|y| < 1 \\ (C/2^{k+1})|y|^\alpha < t \leqslant (C/2^k)|y|^\alpha}} \left(\frac{t}{|y| + t} \right)^{\lambda} \frac{dy\, dt}{t^2} \\ &\leqslant C' \int_{\substack{|y| < 1 \\ (C/2^{k+1})|y|^\alpha < t \leqslant (C/2^k)|y|^\alpha}} \left(\frac{t}{|y|} \right)^{\lambda} \frac{dy\, dt}{t^2} \\ &\leqslant C' \int_{|y < 2} |y|^{-\lambda + \alpha(\lambda - 1)}\, dy \leqslant C' \end{aligned}$$

since $\alpha, \lambda > 1$. (Here and in what follows the value of C' may change in insignificant ways from line to line.)

Next, in the region $S_k^{(2)}$, note that $|\bar{x}|/2 < |\bar{x} - y| < 3|\bar{x}|$, so

$$\begin{aligned} \int_{S_k^{(2)}} \left(\frac{t}{|\bar{x} - y| + t} \right)^{\lambda} \frac{dy\, dt}{t^2} &\leqslant \frac{C'}{|\bar{x}|^\lambda} \int_{|y| < 2|\bar{x}|} \int_0^{(C/2^k)|y|^\alpha} t^{\lambda - 2}\, dt\, dy \\ &\leqslant C' |x|^{\alpha(\lambda - 1) + 1 - \lambda} \leqslant C'. \end{aligned}$$

Finally, we estimate

$$\int_{S_k^{(3)}} \left(\frac{t}{|\bar{x}-y|+t}\right)^{\lambda} \frac{dt\,dy}{t^2}.$$

If $(y, t) \in S_k^{(3)}$ then $|\bar{x}|/2 < |y| < \frac{3}{2}|\bar{x}|$ and $c'(C/2^{k+1})\,|\bar{x}|^{\alpha} < t \leqslant c'(C/2^k)\,|\bar{x}|^{\alpha}$. Then

$$\begin{aligned}
&\int_{S_k^{(3)}} \left(\frac{t}{|\bar{x}-y|+t}\right)^{\lambda} \frac{dy\,dt}{t^2} \\
&\quad \leqslant C' \left\{ \int_{S_k^{(3)} \cap \{|\bar{x}-y| < (c'/2^k)\,|\bar{x}|^{\alpha}\}} \frac{dy\,dt}{t^2} \right. \\
&\qquad \left. + \int_{S_k^{(3)} \cap \{|\bar{x}-y| \geqslant (c'/2^k)\,|\bar{x}|^{\alpha}\}} \left(\frac{t}{|\bar{x}-y|}\right)^{\lambda} \frac{dy\,dt}{t^2} \right\} \\
&\quad = C'(I+II).
\end{aligned}$$

We see that

$$I \leqslant C' \left(\frac{|\bar{x}|^{\alpha}}{2^k}\right)^{-2} \int_0^{|\bar{x}|^{\alpha}/2^k} dt \int_{|\bar{x}-y| < c'\,|\bar{x}|^{\alpha}/2^k} dy \leqslant C',$$

while

$$II \leqslant C' \left(\frac{|\bar{x}|^{\alpha}}{2^k}\right)^{\lambda-2} \int_{c'\,|\bar{x}|^{\alpha}/2^k < |\bar{x}-y| \leqslant 1} |\bar{x}-y| - \lambda \int_0^{c'\,|\bar{x}|^{\alpha}/2^k} dt\,dy \leqslant C'.$$

This proves Lemma 5.3.

Now we proceed with the proof of Theorem 5.2. We construct the required function on the unit circle as follows: Refer back to Theorem 3.1, part (ii), and [2] where this construction is carried out in some detail. We take Φ: $\mathbb{R}^2_+ \to \mathbb{R}^2_+$ given by $\Phi(x, t) = |(x, t)|^{\beta}\,(x, t)$, where $\beta > -1$. Then if $\Delta u = 0$ on $\mathbb{R}^2_+$, it follows that $\operatorname{div}(B\nabla(u \circ \Phi)) = 0$ where $B(x, t)$ is C^{∞} away from 0 and homogeneous of degree 0. The harmonic measure of $\operatorname{div}(B\,\nabla u)$ is homogeneous of degree β, and if $\beta < 0$ it follows that this measure is not in every class $B^p(d\sigma)$ for all $p < \infty$. The coefficients are constant on the real axis, so by a linear change of variables we arrive at an operator $Lu = \operatorname{div}(A\,\nabla u)$ where harmonic measure is not in every B^p class, and whose coefficient function A satisfies $A(x, 0) \equiv I$ for all $x \in \mathbb{R}$. Just as in Theorem 3.1, part (ii), we further perturb L to $\tilde{L}u = \operatorname{div}(\tilde{A}\,\nabla u)$ by setting $\tilde{A} = A$ except in a cone $\Gamma_{\gamma}(0)$ of small aperture γ and setting $\tilde{A} \equiv I$ in $\Gamma_{\gamma}(0)$. If γ is sufficiently small, then Dahlberg's small perturbation theorem implies that the harmonic measure on $\tilde{L}$ is not in every B^p class, while the coefficient $\tilde{A}$ is homogeneous of degree zero (with respect to the usual dilations) and its restriction to S^1 is the function we seek here. This restriction, R, has the properties

(a) $R \in C^\infty(S^1)$,

(b) $R(1,0) = R(-1,0) = I$, and

(c) $R(x,t) \equiv I$ for all $(x,t) \in S^1$ sufficiently close to the point $(0,1)$.

We now show that if $\beta < 0$ and $|\beta|$ is sufficiently small in our construction of R, then if R is extended to be homogeneous of degree 0 with respect to parabolic dilations of order $\alpha > 1$, then the corresponding operator has its harmonic measure in $B^\infty(d\sigma)$.

To see this, call the extension of R a function which is homogeneous of degree 0 with respect to parabolic dilations of order α, $A_\alpha(x,t)$. Let $L_\alpha u = \operatorname{div}(A_\alpha \nabla u)$ in D. Set $a_\alpha(z) = \operatorname{ess\,sup}_{y \in B(z;\,\delta(z)/C)} |A_\alpha(y) - I|$. We claim that for $\lambda > 1$, $g_\lambda^{(1)}(a_\alpha) \in L^\infty$. If we then take $\beta < 0$, $|\beta|$ small enough, it is easy to see that this forces $\|a_\alpha\|_{L^\infty}$ to be as small as we wish, and Theorem 3.2(ii) applies to show that $\omega_{L_\alpha} \in B^\infty(d\sigma)$, finishing the proof of Theorem 5.2. To see that for $\lambda > 1$, $g_\lambda^{(1)}(a_\alpha) \in L^\infty$, we observe that in the regions

$$S_k = \left\{ (y,t) \in B_+ \,\middle|\, \frac{C_1\,|y|^\alpha}{2^{k+1}} < t \leqslant \frac{C_1\,|y|^\alpha}{2^k} \right\}, \qquad k = 1, 2, 3, \ldots,$$

the function $a_\alpha(z) = O(2^{-k})$ and $a(z) \equiv 0$ in $B_+ - \bigcup_{k \geqslant 1} S_k$ provided C and C_1 are large enough. Therefore, if $\lambda > 1$, by Lemma 5.3 we have

$$\begin{aligned} \|g_\lambda^{(1)}(a_\alpha)\|_\infty &\leqslant \sum_{k \geqslant 1} \|g_\lambda^{(1)}(a_\alpha \chi_{S_k})\|_\infty \\ &\leqslant C' \sum_{k \geqslant 1} \|a_\alpha \chi_{S_k}\|_\infty \leqslant C' \sum_{k \geqslant 1} 2^{-k} \leqslant C'. \end{aligned}$$

This concludes the proof of Theorem 5.2.

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